Finite size scaling of the balls in boxes model

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Abstract

We discuss the finite size behaviour in the canonical ensemble of the balls in boxes model. We compare theoretical predictions and numerical results for the finite size scaling of cumulants of the energy distribution in the canonical ensemble and perform a detailed analysis of the first and third order phase transitions which appear for different parameter values in the model.

Introduction

The balls in boxes model is a simple statistical model describing an ensemble of balls distributed in boxes subject only to a single global constraint. In some circumstances the model undergoes a phase transition driven by a condensation of the balls into a single box.

The model has been introduced as a mean field approximation to lattice gravity [1, 2]. Despite its simplicity, the model captures the main features of the phase transition observed in lattice Euclidean quantum gravity models (*ie* dynamical triangulations) such as the discontinuity of the transition [3, 4] and the appearance of singular structures [5, 6, 7, 8].

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Variations of the phase transition of the balls in boxes type can be found in many areas of physics like the thermodynamics of hadrons or strings [9, 10], branched polymers [11, 12] or percolation[1]. The balls in boxes model is also closely related to urn models [13, 14] and the spherical model [15]. It is thus a rather generic type of phase transition.

The model has an interesting phase structure : by tuning a single parameter one can change the order of the transition or make the transition disappear entirely [2, 16]. This sort of phase transition was discovered in the spherical model by Berlin and Kac [15]. The nature of the transition is quite different from more standard phase transitions in the theory of critical phenomena which result from the appearance of long range spatial correlations. Here the transition does not refer to any correlations in space. Instead, it is a kinematic condensation which comes about when a fugacity of the series representing the partition function hits the radius of convergence of the series. This fugacity sticks to the radius of convergence and cannot move. At this point the corresponding physical system enters the condensed phase. From this point of view the mechanism of the transition is very similar to Bose–Einstein condensation. The basic difference between the balls in boxes type of condensation and the Bose-Einstein one is that in the latter the system condenses into an energetically favoured ground state, while in the former balls condense into a box which is indistinguishable from the remaining ones. The balls in box condensation thus spontaneously breaks the permutation symmetry of the boxes.

The thermodynamic limit of the balls in boxes model and the relation to lattice gravity has been discussed in a series of papers [1, 2, 16]. Here we discuss finite size effects and confront theoretical predictions with the numerical analysis. The numerical analysis is done using an exact algorithm which recursively generates partition functions for systems of moderate size *ie* up to a few thousand boxes.

The model and thermodynamic limit

In this section we briefly recall the model [2, 16]. The model describes N balls distributed in M boxes. Each box has a certain weight $p(q)$ which depends only on the number q, of balls in it. The weights $p(q)$ are the same for all boxes. For a given partition of balls in boxes $\{q_1, q_2, \ldots, q_M\}$, the total weight of the configuration is equal to the product $p(q_1) \ldots p(q_M)$ – *ie* the

[1]The appearance of a percolating cluster in the percolation theory can be translated into the condensation of balls in one box, which after condensation contains extensive number of balls.

boxes are almost independent. They are not entirely independent because of the constraint $q_1 + \cdots + q_M = N$. It is this constraint which makes the model nontrivial. The model is defined as a sum over all partitions of N balls in M boxes weighted by the product weight. This is a fixed density ensemble, with density $\rho = N/M$. In the thermodynamic limit $M \to \infty$, and $\rho = const$, the behaviour of the system depends on the density ρ. For some choices of weights $p(q)$, there exists a critical value ρ_{cr} above which the condensation of balls takes place, which means that a box appears in the system which contains a finite fraction of the balls. The phase transition in ρ corresponding to this condensation is of third or higher order [2].

The fixed ρ ensemble is a kind of microcanonical ensemble. One can also consider ensembles with varying density. There are two natural candidates for such "canonical" ensembles : for a fixed M one can allow for varying N by introducing a conjugate coupling to N. In this formulation boxes decouple and we are left with M copies of the urn model. Alternatively for fixed N one can allow for varying M. This corresponds to the ensembles studied in lattice gravity [1] and will be the subject of our investigations.

The partition function for this canonical ensemble reads :

$$Z(N,\kappa) = \sum_{M=1}^{\infty} e^{\kappa M} \sum_{q_1,\cdots,q_M} p(q_1)\cdots p(q_M)\ \delta_{q_1+\cdots+q_M,N} \tag{1}$$

$$= \sum_{M=1}^{\infty} e^{\kappa M} z(N,M)\,. \tag{2}$$

The canonical function can formally be treated as a discrete Laplace transform of the microcanonical partition function $z(N,M)$ as the second line of the equation above shows. Analogously, in the same language, we can introduce the grand–canonical partition function as the Laplace transform of the canonical one :

$$\mathcal{Z}(\mu,\kappa) = \sum_{N=1}^{\infty} e^{-\mu N} Z(N,\kappa)\,. \tag{3}$$

Each box is assumed to contain at least one ball. Therefore the sum in the last equation starts from $N = 1$.

The weights $p(q)$ are a priori independent parameters that we assume to be non–negative. In order that the large N limit be well defined the weights cannot grow faster than exponentially with q. On the other hand the exponential growth factor is irrelevant from the point of view of the critical behaviour of the model, as can be seen below. Namely, if one changes weights

as :

$$p(q) \to p'(q) = e^{-\kappa_0} e^{\mu_0 q} p(q)\,, \tag{4}$$

where μ_0 and κ_0 are some constants, the partition functions change as :

$$\begin{aligned} z(N,M) \to z'(N,M) &= e^{\mu_0 N - \kappa_0 M} z(N,M) \\ Z(N,\kappa) \to Z'(N,\kappa) &= e^{\mu_0 N} Z(N,\kappa-\kappa_0) \\ \mathcal{Z}(\mu,\kappa) \to \mathcal{Z}'(\mu,\kappa) &= \mathcal{Z}(\mu-\mu_0,\kappa-\kappa_0)\,. \end{aligned} \tag{5}$$

Thus the change of the exponential factor in the weights merely causes a redefinition of the coupling constants μ and κ. So from the point of view of the phase structure of the model, only the sub-exponential factors of weights matter. Here we restrict ourselves to the power like weights :

$$p(q) = q^{-\beta}\,, \quad q = 1, 2, \dots\,, \tag{6}$$

which capture a variety of essential phase transitions in the model. Some remarks on the behaviour for weights of general form can be found in [16].

In the large N limit the model can be solved analytically [16]. The free energy :

$$\phi(\kappa) = \lim_{N\to\infty} \frac{1}{N} \log Z(N,\kappa) \tag{7}$$

has a singularity in κ at $\kappa_{cr} = -\log\zeta(\beta)$, for $\beta \in (1,\infty)$ and has no singularity otherwise. $\zeta(\beta)$ is the Riemann Zeta function. The phase for $\kappa > \kappa_{cr}$ is called fluid, and for $\kappa < \kappa_{cr}$ condensed.

The order parameter for the transition is the first derivative of the free energy with respect to κ which corresponds to the inverse average density of balls per box :

$$r = \phi^{(1)}(\kappa) = \frac{\partial\phi}{\partial\kappa} = \frac{\langle M\rangle}{N}\,. \tag{8}$$

It vanishes in the condensed phase *ie* for $\kappa < \kappa_{cr}$. The average $\langle \dots \rangle$ is taken with respect to the canonical ensemble (2).

The order of the transition depends on β. For β in the range $\beta \in (2,\infty)$ the transition is first order. When $\kappa \to \kappa_{cr}^{+}$ approaches the critical point the order parameter goes to a nonzero constant $r_{disc} = \zeta(\beta)/\zeta(\beta-1)$ corresponding to the discontinuity of the order parameter at the critical point. If one treats M in the partition function (2) as the energy of the system, then the discontinuity of r corresponds to the latent heat. For $\beta \to 2$ the latent

heat vanishes and the transition becomes continuous. For β in the range $\beta \in (1, 2]$, r has a branch point singularity at κ_{cr}. In this case, the order parameter vanishes when one approaches the critical point $\Delta\kappa = \kappa - \kappa_{cr} \to 0^+$:

$$r = \phi^{(1)} \sim \Delta\kappa^{x_1}, \quad \text{where} \quad x_1 = \frac{2-\beta}{\beta-1} \quad . \tag{9}$$

The transition is n-th order for $\beta \in (\frac{n+1}{n}, \frac{n}{n-1}]$. When β approaches one ($\beta \to 1^+$) the transition becomes softer and softer ($n \to \infty$) and eventually disappears at $\beta = 1$. For β smaller than one the system has only a fluid phase and displays no phase transition.

Finite size scaling

Let us denote the most singular term in the free energy by $\phi_{sing}(\kappa)$ and the related exponent by x_0 :

$$\phi_{sing} \sim \Delta\kappa^{x_0} \, . \tag{10}$$

To match it with the scaling (9) for $\beta \in (1, 2)$, we must have :

$$x_0 = \frac{1}{\beta - 1} \, . \tag{11}$$

For large but finite N one expects that this behaviour gets substituted by the double scaling law with two exponents which we denote by A_0 and B(see [18] and reference therein) :

$$\phi_{sing}(\Delta\kappa, N) = N^{A_0} f(\Delta\kappa N^B) \, , \tag{12}$$

where $f(\xi)$ is a certain universal function of one argument $\xi = \Delta\kappa N^B$. When $\Delta\kappa$ is fixed and N goes to infinity one expects to asymptotically recover the singularity of the thermodynamic limit $\phi_{sing}(\Delta\kappa, N \to \infty) \sim \Delta\kappa^{x_0}$ independent of N. This means that the universal function must have the following large ξ behaviour : $f(\xi) \sim \xi^{x_0}$ and :

$$x_0 = -A_0/B \, . \tag{13}$$

The singular part of the free energy is mixed with less singular and analytic parts. While for $\Delta\kappa \to 0$ the ϕ_{sing} will eventually dominate over less singular parts it can still itself be dominated by the analytic part. It is therefore difficult to extract the scaling part ϕ_{sing} directly from the function ϕ. Instead, one does it by calculating the n-th derivative of ϕ for n large enough

that the exponent x_n is negative. We use the convention that the exponent x_n is the power $\phi^{(n)}_{sing}(\kappa) \sim \Delta\kappa^{x_n}$ of the most singular part of the n-th derivative. Thus we trivially have $x_n = x_0 - n$. The first value of $n = 1, 2 \ldots$ for which x_n is negative, gives the order of the transition. For this n, the singular part of the n-th derivative blows up at κ_{cr} and dominates the analytic part. Therefore in this case one can skip the subscript *sing* when writing a scaling formula analogous to (12) for divergent derivatives :

$$\phi^{(n)}(\Delta\kappa, N) = N^{A_n} f^{(n)}(\Delta\kappa N^B) + \ldots , \quad \text{with} \quad A_n = nB + A_0 \tag{14}$$

keeping in mind that there are corrections, (denoted by dots) which may be important for finite N. Asymptotically, for $N \to \infty$ they are, however, negligible in comparison to the displayed part.

The knowledge of the exponent x_0 does not suffice to calculate A_0 and B. In the standard finite size scaling analysis the value of the exponent B can be obtained by a simple argument [17, 18]. There are two relevant length scales in a d–dimensional system undergoing a continuous phase transition : the linear extension $L = N^{1/d}$ and the correlation length ξ, diverging as $\xi \sim (\Delta T)^{-\nu}$, when $\Delta T = T - T_{cr}$ goes to zero. The critical behaviour sets in when the lengths become comparable :

$$L \sim \xi \tag{15}$$

giving :

$$\Delta T \sim N^{-\frac{1}{d\nu}} \quad \longrightarrow \quad B = \frac{1}{d\nu} \, . \tag{16}$$

Alternatively one can say that critical properties of the system depend on the dimensionless ratio : ξ/L. We use here a similar reasoning. Namely, we extract characteristic scales from the distribution of the box occupancy number $\pi(q)$ *ie* the probability that a box has q balls :

$$\pi(q) = \left\langle \frac{1}{M} \sum_{i=1}^{M} \delta(q_i - q) \right\rangle = \frac{p(q) e^{\kappa} Z(N-q, \kappa)}{Z(N, \kappa)} \, . \tag{17}$$

In the large N limit the distribution has the form :

$$\pi(q) \sim p(q) \, e^{-\phi(\Delta\kappa) q} \tag{18}$$

as can be seen from (7). There are two scales which govern this distribution when the critical point is approached from the fluid phase : the damping scale factor $[\phi(\Delta\kappa)]^{-1}$ and the system size N. Thus, in analogy to (15) one

expects that the dimensionless combination of the scales which defines universal critical properties of the system is : $[\phi(\Delta\kappa)]^{-1}/N$. Using the relation (10) we eventually get :

$$B = x_0^{-1} = \beta - 1 \quad \text{and} \quad A_0 = -1\,. \tag{19}$$

Inserting this into (12) we obtain $A_n = nB - 1$. This is a sort of Fisher scaling relation[2].

There are some thermodynamic inequalities which follow directly from the Fisher relation. Firstly, one can see that B must be greater than zero, $B > 0$, in order that there exist n such that $A_n > 0$. Otherwise there were no transition. This inequality is in accordance with the inequality $\beta > 1$ which gives the condition for existence of the transition in this model, as discussed in the text after the equation (9). Secondly, $A_2 = 2B - 1$ must be smaller than one $A_2 < 1$ which amounts to $B \leq 1$. This is because the second cumulant may not grow faster than $\frac{1}{4}N$ as can be seen from (22) by taking into account the fact that values of M lie in the range $[1, N]$. In fact the limiting value $B = 1$ corresponds to the first order transition scaling which follows from the presence of a non–vanishing latent heat[3] *ie* $r_{disc} > 0$. This holds for all $\beta \leq 2$. In this range of β the exponents $A_2 = B = 1$ are constant, while the discontinuity r_{disc} changes with β.

Formula (14) gives a practical way of computing the most relevant singularity of the partition function from the finite size analysis of numerical data. Namely, one computes divergent derivatives, for them estimates the exponents A_n, B and using the relation $A_n = A_0 + nB$ one calculates :

$$x_n = -A_n/B = x_0 + n\,. \tag{20}$$

Numerically, derivatives $\phi^{(n)}$ of the free energy are computed as cumulants of the canonical distribution for M (2) :

$$\phi^{(1)} = \frac{\langle M\rangle}{N} \tag{21}$$

$$\phi^{(2)} = \frac{\langle M^2\rangle - \langle M\rangle^2}{N} \tag{22}$$

$$\phi^{(3)} = \frac{\langle M^3\rangle - 3\langle M^2\rangle\langle M\rangle + 2\langle M\rangle^2}{N}\,. \tag{23}$$

$$\ldots$$

[2]In the standard considerations of the second order phase transition in the theory of critical phenomena the exponent $B = 1/\nu d$ and $A_2 = \alpha/\nu d$, The exponent α is the heat capacity exponent *ie* the second cumulant of energy. Thus, the relation $A_2 = 2B - 1$ corresponds to $\alpha = 2 - \nu d$, known as Josephson's or Fisher's law.

[3]In the standard theory of critical phenomena $B = 1/\nu d = 1$ gives the canonical exponent $\nu = 1/d$.

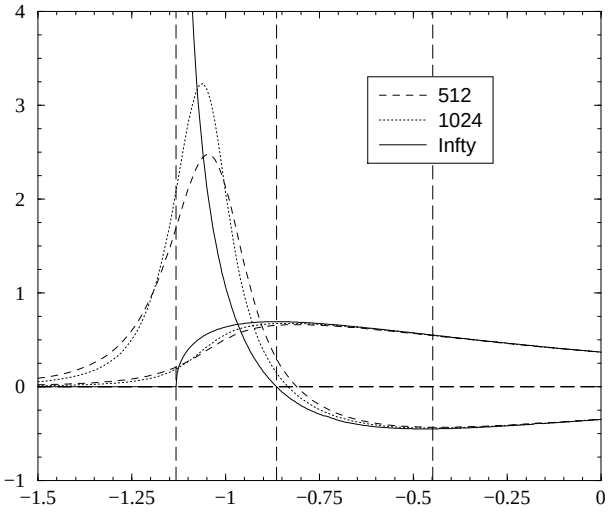

Figure 1: The second and third derivatives of $\phi(\kappa, N)$ at $\beta = 1.4$ in thermodynamical limit and for two different system sizes. The critical value is at $\kappa_{cr} \approx -1.133$, the position of the maximum of the second cumulant is at $\kappa_{max} \approx -0.865$, and the minimum of the third one at $\kappa_{min} \approx -0.449$. Those are indicated by dashed vertical lines.

For finite N the first cumulant $\phi^{(1)}(\kappa, N)$ grows monotonically from zero to one as a function of κ, while the higher order cumulants $\phi^{(n)}$ have $n-1$ extrema. If a cumulant is divergent, the leftmost maximum corresponds to the scaling, critical part described by the formula (14). The remaining extrema lie in the fluid phase and come from the non-scaling part. In the thermodynamic limit the leftmost maximum of a divergent cumulant approaches the singularity : $\phi^{(n)}(\kappa) \sim \Delta\kappa^{x_n}$.

As an example consider the figure 1 where derivatives of $\phi(\kappa, N)$ are plotted for $\beta = 1.4$. In this case system undergoes third order phase transition at $\kappa_{cr} = -\log\zeta(1.4)$. When κ approaches the critical value $\Delta\kappa = \kappa - \kappa_{cr} \to 0^+$ the first and second derivatives of $\phi(\kappa)$ vanish as $\phi^{(1)} \sim \Delta\kappa^{3/2}$, $\phi^{(2)} \sim \Delta\kappa^{1/2}$, whereas the third one diverges as $\phi^{(3)} \sim \Delta\kappa^{-1/2}$. The second derivative has a maximum that lies in the fluid phase far from the transition (1). The third derivative has a minimum in the fluid phase away from the critical value κ_{cr}. For finite size the third derivative $\phi^{(3)}(\kappa, N)$ has a maximum and a minimum. When N goes to infinity, the position of the maximum tends to κ_{cr} and its height grows according to a finite size scaling (14). The position of the minimum approaches a value κ_{min} which lies in the fluid phase far from the critical region.

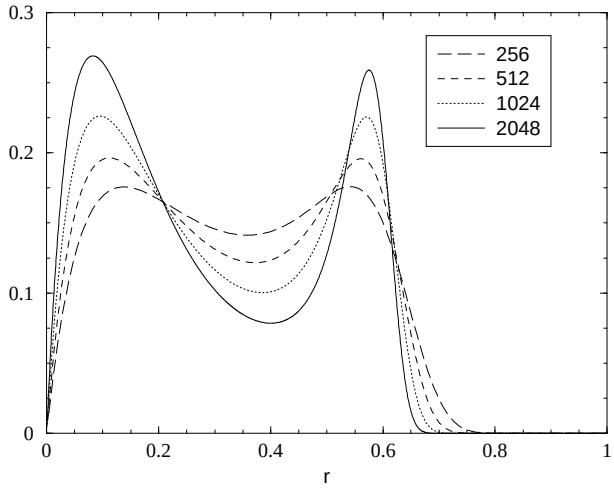

Figure 2: The probability distributions of the energy $r = M/N$ for pseudo-critical values of $\kappa_{cr}(N)$ at $\beta = 2.6$ for $N = 256, 512, 1024, 2048$.

For higher order cumulants the number of such noncritical extrema increases with the order of the derivative. It may even happen that for a given system size the height of a non critical maximum is larger than of the critical one. Generally, to determine A_n and B one should analyze only the scaling of the leftmost maximum of a cumulant, where the information about the singularity of $\phi^{(n)}$ is encoded.

In the next sections we will present a numerical analysis of the finite size data for the different ranges of β. Before doing this, let us briefly describe the algorithm to generate the finite size partition function is $Z(N, \kappa)$

We calculate the partition function $Z(N, \kappa)$ in two steps. First we compute values of the microcanonical partition function $z(N, M)$ for all M in the range $[1, N]$ by the following recurrence relation:

$$z(N, M) = \sum_{q=1}^{N} z(N - q, M - 1)p(q), \tag{24}$$

with the initial condition $z(1, q) = q^{-\beta}$. Inserting the numbers $z(N, M)$ directly to the definition (2) we obtain $Z(N, \kappa)$. The maximal size N which can be reached by this procedure is a few thousand. It is limited by floating point instabilities accumulated in the recurrence relation (24). To test the stability we check whether the results stay intact (up to a shift in κ) under the change of weights (4).

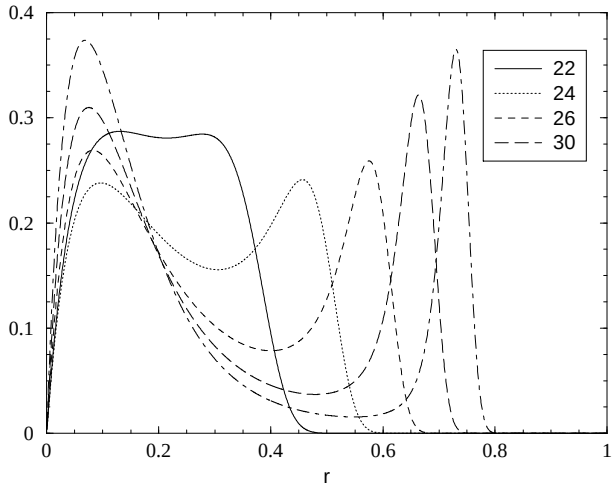

Figure 3: The probability distributions of the energy $r = M/N$ for pseudocritical values of $\kappa_{cr}(N)$ for $N = 2048$ at $\beta = 2.2, 2, 4, 2, 6, 2.8, 3.0$.

Finite size analysis results

First order phase transition

We start the finite size analysis with the range $\beta > 2$. According to the discussion above, the phase transition should be first order. A typical signal of first order phase transition is a double peak in the distribution of the energy corresponding to the coexistence of two phases at the transition with the latent heat being the separation between peaks. Let us look for this signal in our case. In figure 2 we plot the pseudocritical distributions of the energy density $r = M/N$ for different system sizes N for $\beta = 2.6$ for values of $\kappa_{cr}(N)$ for which the both peaks have equal heights. These values can be taken as pseudocritical ones. The position of the left peak goes to zero when N goes to infinity, while of the right one to $r_{disc}(\beta) = \zeta(\beta)/\zeta(\beta - 1) = 0.571$ known from the analytic calculations in the thermodynamic limit. When β changes the position of the right peak moves (figure 3). For large β, the discontinuity r_{disc} goes to one, while for $\beta \to 2$ it disappears and the two peaks merge. The depth of the valley between the peaks increases with N which means that the configurations which do not belong to either of phases become more and more suppressed. The suppression becomes more visible for larger β (figure 3). On the contrary, for β close to 2 the valley is small or even absent. In this case the size of the system must be increased sufficiently for the valley to be visible.

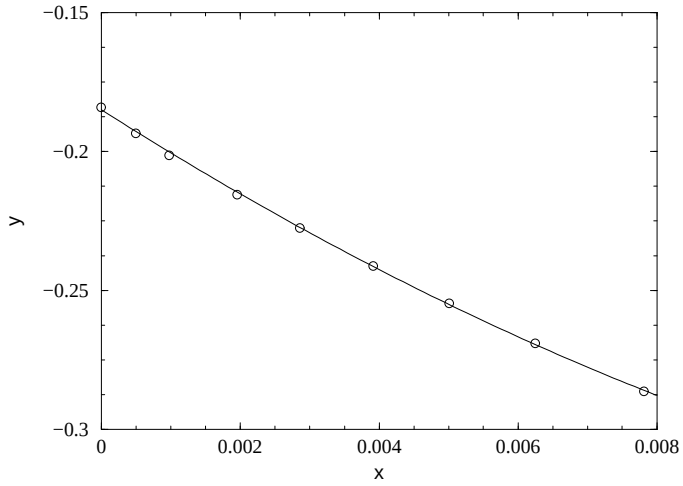

Figure 4: The pseudocritical value $\kappa_{cr}(N)$ versus $1/N$ for $\beta = 3.0$. The curve going through the data points corresponds to the best fit $\kappa = \kappa_{cr} + a/N\ (1 + b/N)$.

The range of κ for which two peaks coexist is called pseudocritical region. For κ in this region the relative heights of the peaks vary. The extent of the region is inversely proportional to the size of the system, N. Outside the pseudocritical region only one peak survives. If κ changes within the pseudocritical region, the average of the distribution $r = \langle M \rangle / N$ moves very quickly between two peaks. The slope of the curve $r(\kappa)$ grows linearly with N, and eventually becomes infinite when N is sent to infinity, leading to the discontinuity r_{disc} at κ_{cr}. The slope of the curve corresponds to the second derivative, which is heat capacity. Because the heat capacity grows linearly with N, the system has latent heat corresponding to the energy needed to move states of the system from one peak to the other. Another characteristic signal of first order phase transition which can be read off from the formula (14) is that the position of the maximum of the second cumulant should asymptotically lie on a curve $\Delta\kappa N = const$, which means that $\kappa(N) = \kappa_{cr} + const/N$. As an example we show in figure 4 the behaviour of $\kappa_{cr}(N)$ versus N. The data points are fitted to the formula with next-to-leading corrections of the standard form : $\kappa(N) = \kappa_{cr} + \frac{a}{N}(1 + \frac{b}{N})$. The fit gives[4] $\kappa_{cr} = -0.185(1)$, $a = -15.9(3)$ and $b = -24(2)$. The coefficient b of

[4]To estimate the errors of the fit parameters we use the following procedure. The data consists of n points for different volumes. We successively omit one of them and fit the formula to the remaining ones obtaining n different fits. Having done this, for each parameter of the fit we have a distribution of n values. We find the average and the

the correction term is large. Skipping the correction b/N in the fit would decrease the quality of the fit and would significantly change the estimate of the κ_{cr} which agrees with the analytic result $\kappa_{cr} = -\log\zeta(3) = -0.184$ with the correction term.

To summarize, the standard signals and finite size scaling characteristic of a first order transition are observed in the range $\beta \in (2, \infty)$ as expected.

Continuous phase transition

For $\beta \in (3/2, 2)$ the linear growth of the maximum of the second cumulant of the first order phase transition changes to a sublinear behaviour $\sim N^{A_2}$, where $0 < A_2 < 1$, corresponding to a second order phase transition. There is no double peak signal in the distribution of M. The pseudocritical point is defined by the value of κ at which the second cumulant of the distribution of M is the largest. The second order phase transitions have been extensively analyzed numerically in a number of papers. Therefore, we prefer here to go directly to the third order phase transition, $\beta \in (4/3, 3/2)$, where some new ingredients like the presents of non-scaling extrema appear on top of the standard finite size effects known from second order transitions. As an example, we consider the case $\beta = 1.4$ mentioned before, for which we expect that the second cumulant does not diverge, while the third one does with the exponent $x_3 = -1/2$. As discussed before, we will concentrate our attention on the leftmost maximum of the third cumulant, which signals the appearance of the divergence in the thermodynamic limit. For N going to infinity, the height of the maximum is expected to grow as a power N^{A_3} (14), possibly with some deviations for finite N coming from non-scaling part denoted by dots in (14). Indeed, it turns out that in the range of N from 16 to 4096 which we have covered by the recursive method (24) the corrections to the asymptotic formula are strong, as can be seen in figure 5, where the height of the maximum of the third cumulant versus N in the log-log scale is plotted. This is clearly non–linear and we define the effective exponent $A_3(N)$ as the slope of the line fitted through three consecutive points with N standing for the biggest size N of the three points used in the linear fit. The effective exponent is clearly far from reaching an asymptotic N–independent value (figure 6). To be able to find the asymptotic value $A_3^* = A_3(N \to \infty)$ from this data, one would have to know the form of subleading terms. Since we do not, we propose here a phenomenological approach. We postulate a

width of the distributions which we take as the mean and the error of the parameters. In doing this we assume that all the data points are equally important, which is the simplest possible assumption. This procedure is not a statistical analysis, but merely a way of presenting data.

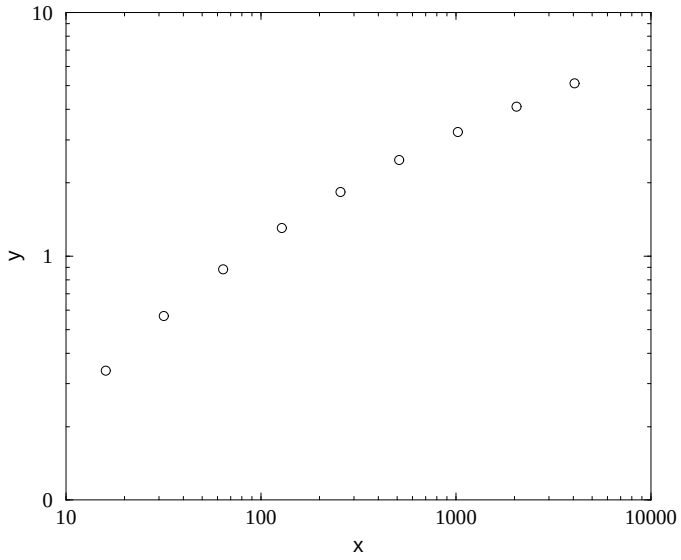

Figure 5: The data points represent the maximum of the third cumulant for different N. They are plotted in a log-log scale.

phenomenological form of corrections to the running effective exponent

$$A_3(N) = A_3^* + bN^{-c}. \tag{25}$$

It turns out, that this formula gives a good fit (figure 6) with the values $A_3^* = 0.16(1)$, $b = 1.58(1)$ and $c = -0.27(1)$, estimated by the same procedure as discussed in the footnote on one of earlier pages. From the fit one can make a qualitative estimate of the subleading corrections. A simple calculation shows that N must be of order of 10^6 in order that the running exponent differ from the asymptotic value A_3^*, say, by 0.04. It is of course a rough estimate, but it points to how slowly the corrections change with N.

The standard method of measuring the exponent B relies on tracing the position of the maximum for different sizes. For large N it approaches an asymptotic value. This value corresponds to the critical temperature. The exponent in the function which measures the distance of the position of the maximum from the critical temperature is proportional to $\Delta\kappa \sim N^{-B}$. This gives the possibility of computng B. Due to the strong subleading corrections, this method does not give a good estimate in our case. Instead we propose another one. Unlike in the standard Monte Carlo, in our case the cost of computing cumulants does not grow with the order of the cumulant. We use this fact, to compute the growth exponents A_n for higher order – divergent cumulants, $n = 3, 4 \ldots$. Then we use the formula for linear growth :

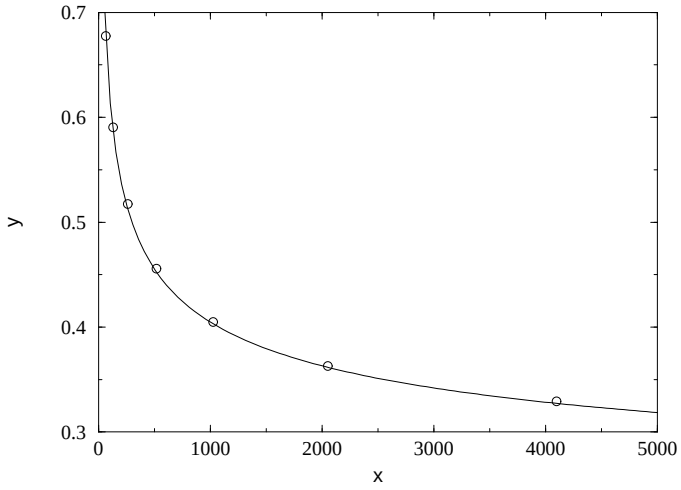

Figure 6: The effective exponent $A_3(N)$ calculated as a slope of the lines in the previous figure.

$A_n = A_0 + nB$, which allows us to compute the exponent B as the slope of the line. As an input for A_n we take the asymptotic values A_n^* with the errors, for $n = 3, 4, 5, 6$ estimated by the method discussed above. The fit is shown in figure 7. It gives : $A_0 = -1.005(42)$, $B = 0.387(12)$ and $\chi^2/dof = 0.45$. The errors quoted correspond to the 99.7% confidence level[5]. The deviation from the theoretical values $A_0 = -1.0$ and $B = 0.4$ predicted from the Fisher relation are rather small : less than 1% for A_0 and 4% for B . The former value agrees within the errors with the theoretical one, while the latter one is a little outside the error bars.

However, when one inserts those values to the formula $x_3 = -A_0/B - 3$ and assumes the maximal correlation between A_0 and B, the errors are strongly enhanced and one obtains $x_3 = -0.403(40)$. This is 20% off from the theoretical value $x_3 = -0.5$ and twice as much beyond the estimated errors. The assumption about the strong correlation is dictated by the covariance matrix for the fit. On the other hand, if we instead assume that the Fisher relation indeed holds, as suggested by both the analytical and experimental evidence, we can set $A_0 = -1$ and in this case we have only one parameter relation $x_3 = 1/B - 3$ which leads to $x_3 = -0.416(80)$.

In interpreting the results one has to remember that they depend on

[5]For a cross-check we have also done a fit assuming equal errors on each A_n and using condition $\xi^2/dof = 1$. The results $A_0 = -1.017(48)$ and $B = 0.389(10)$ are consistent with the ones obtained before.

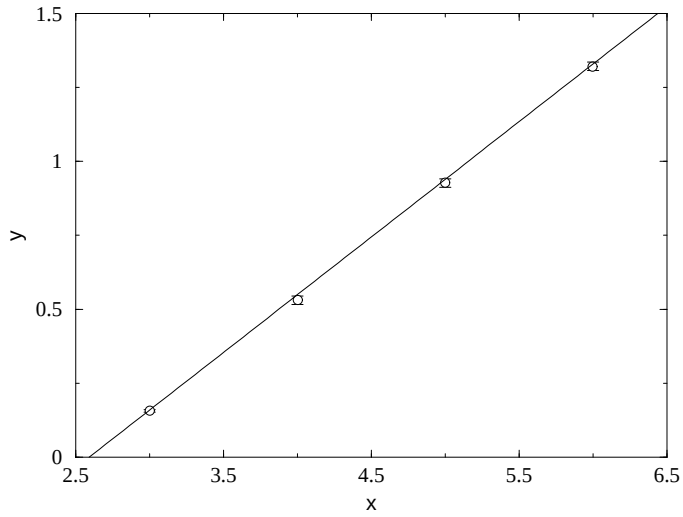

Figure 7: The linear fit of the exponents A_n for $n = 3, 4, 5, 6$. The slope of the line corresponds to the exponent B.

the phenomenological fit (25) extrapolated over several orders of magnitude. Such a fit gives meaningful results only due to the fact, that the small volume data points are *exact*, but on the other hand one has to keep in mind that in general such a procedure may introduce a systematic error which is hard to quantify.

Reduction of the error would require either a knowledge of the form of the next-to-leading correction or pushing the computations to the system sizes of a few orders of magnitude larger.

Discussion

The balls in boxes model provides a useful laboratory for testing finite size scaling. On one hand one can predict critical exponents theoretically, on the other hand one can exactly calculate the finite size partition function. In this respect the model is exceptional, since most of the models must rely on noisy Monte Carlo data. A comparison of the asymptotic form (14) and the finite size results indicates the presence of strong corrections to scaling. The corrections come from the analytic and less singular part of the partition function. The corrections seem to be a very slowly varying function of the system size as figure 6 demonstrates. The phenomenological approach employed to estimate the next–to–leading corrections and the limiting asymptotic value of

the effective exponent is possible because unlike the Monte Carlo data, our data points have no statistical uncertainties.

The finite size analysis results support the hypothesis that the critical properties of the model are encoded in the one box probability $\pi(q)$. The universal scaling properties are obtained from comparing two scales present in the model : the inverse of an exponential fall–off parameter of the effective probability distribution $\pi(q)$ and the system size N. The former one measures the extent of fluctuations of the number balls in a box and the latter one serves as a natural cut-off of those fluctuations. As mentioned in the introduction this type of transition plays an important role in many physical systems. In particular it is relevant for understanding the transition in lattice gravity models[6].

The model provides a nice example of the finite size pattern for a first order phase transition. The results for the distribution are exact and easily reproducible so one can measure all relevant quantities and quickly test any new ideas. Thus, we hope, the model may also prove useful for studying generic properties of first order transitions.

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